TENSOR PRODUCT FACTORIZATION AND MULTIPLIERS

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0. INTRODUCTION. Let G be a locally compact Abelian topological group, let S be a Segal algebra on G, and let W be an essential L_1-convolution module. In this note we give a simple proof of a necessary and sufficient condition for the multipliers from S to W^*, the dual space of W, to be topologically isomorphic to W^*. The condition involves whether or not W can be expressed as a certain tensor product of S and itself. We shall also apply this theorem to prove some results about tensor product factorization and about multipliers.

Before we take up these considerations in detail we need to recall some definitions and results that will be required. If G is a locally compact Abelian topological group, then $L_1(G)$ will denote the usual convolution group algebra of G. A Banach space $(W, \|\cdot\|_W)$ is said to be an L_1-module if there exists a multiplication operation between elements of $L_1(G)$ and elements of W, denoted by $\circ$, such that W is an algebraic module over $L_1(G)$ with respect to this multiplication and for which there exists some constant $B_W > 0$ such that $\|f\circ w\|_W \leq B_W \|f\|_1 \, \|w\|_W$ for every $f \in L_1(G)$ and $w \in W$. The symbol $\|\cdot\|_1$ denotes the usual norm in $L_1(G)$. If W is an

L_1-module, then so is the dual space W^* of W provided we define the module somposition of $f \in L_1(G)$ and $w^* \in W^*$ by $\langle w, f\circ w^*\rangle = \langle f\circ w, w^*\rangle$, $w \in W$. An L_1-module W is essential if $W = L_1\circ W = \{f\circ w | f \in L_1(G), w \in W\}$, and it is said to be an L_1-convolution module if the module composition $\circ$ is the usual convolution product $*$.

A Banach subalgebra $(S, \|\cdot\|_S)$ of $L_1(G)$ is said to be a Segal algebra if S is a translation invariant $\|\cdot\|_1$-dense subalgebra of $L_1(G)$ such that for every $g \in S$ the mapping $s \to \tau_s g$ of G to S is continuous and $\|\tau_s g\|_S = \|g\|_S$, $s \in G$. The symbol $\tau_s g$ denotes the translate of g by s, that is, $\tau_s g(t) = g(t-s)$, $t \in G$. It follows from the conditions of the definition that a Segal algebra S is an ideal in $L_1(G)$, that there exist some constant $C > 0$ such that $\|g\|_1 \leq C\|g\|_S$, $g \in S$, and that $\|f * g\|_S \leq \|f\|_1 \|g\|_S$, $f \in L_1(G)$ and $g \in S$. Without loss of generality we may and do assume that $C = 1$. In particular, every Segal algebra is an essential L_1-convolution module, and if G is discrete, then there are no proper Segal algebras in $L_1(G)$. We give next some specific examples of Segal algebras.

(a) Let G be an infinite compact Abelian topological group. Then $C(G)$, the space of continuous complex-valued functions on G, with the supremum norm $\|\cdot\|_\infty$, and the usual L_p-spaces $L_p(G)$, $1 < p < \infty$, are proper Segal algebras.

(b) Let G be a nondiscrete locally compact Abelian topological group, let $\hat{G}$ denote the dual group of G, and let $\hat{f}$ denote the Fourier transform of $f \in L_1(G)$. Then for each p, $1 \leq p < \infty$,

$$A_p(G) = \{f | f \in L_1(G), \hat{f} \in L_p(\hat{G})\}$$

is a proper Segal algebra with the norm

$$\|f\|_{A_p} = \|f\|_1 + \|\hat{f}\|_p.$$

(c) Let G be a nondiscrete locally compact Abelian topological group and let $C_o(G)$ denote the space of continuous complex-valued functions on G that vanish at infinity. Then $L_1(G) \cap C_o(G)$ is a proper Segal algebra with the norm

$$\|f\|_{L_1 \cap C_o} = \|f\|_1 + \|f\|_\infty$$

If $1 < p < \infty$, then $L_1(G) \cap L_p(G)$ is a proper Segal algebra with the norm

$$\|f\|_{L_1 \cap L_p} = \|f\|_1 + \|f\|_p .$$

Examples of L_1-convolution modules that are not Segal algebras are provided, for instance, by $C_o(G)$, G noncompact, and by $L_\infty(G)$, the essentially bounded measurable functions on G. The module $C_o(G)$ is essential, whereas $L_\infty(G)$ is not [3, p. 283]. These and other results concerning Segal algebras can be found in [8, pp. 16-26, 34-38].

If S is a Segal algebra and W is an L_1-convolution module, then we define the linear space $S \underline{\otimes} W$ to be all those $w \in W$ of the form $w = \Sigma_{k=1}^{\infty} g_k * w_k$ where $\{g_k\} \subset S$, $\{w_k\} \subset W$ and $\Sigma_{k=1}^{\infty} \|g_k\|_S \|w_k\|_W < \infty$. The space $S \underline{\otimes} W$ is a Banach space with the norm

$$|||w||| = \inf\{\Sigma_{k=1}^{\infty} \|g_k\|_S \|w_k\|_W \mid w = \Sigma_{k=1}^{\infty} g_k * w_k\}$$

It is easily verified that $\|w\|_W \leq B_W |||w|||$, $w \in S \underline{\otimes} W$. Thus the identity mapping $\iota : S \underline{\otimes} W \to W$ is continuous. The space $S \underline{\otimes} W$ can also be shown to be isometrically isomorphic to the L_1-module tensor product $S \otimes_{L_1} \bar{W}$ of S and $\bar{W}$.

A discussion of this result can be found in [7].

If S is a Segal algebra and W is an L_1-module, then we denote by $\mathrm{Hom}_{L_1}(S,W)$ the Banach space of continuous linear transformations T from S to W such that $T(f * g) = f \circ T(g)$, $f \in L_1(G)$ and $g \in W$. Such transformations are called <u>multipliers</u> or <u>module homomorphisms</u>. If W is an L_1-convolution module, then $\mathrm{Hom}_{L_1}(S, W^*)$ is isometrically isomorphic to $(S \circledast W)^*$ [7, p. 6, 9, p. 461]. This isomorphism β is defined by the equations.

$$\langle w, \beta(T)\rangle = \Sigma_{k=1}^{\infty} \langle w_k, T(g_k)\rangle$$

which are to be valied for every $T \in \mathrm{Hom}_{L_1}(S, W^*)$ and $w = \Sigma_{k=1}^{\infty} g_k * w_k \in S \circledast W$. In the sequel we shall be particularly interested in the case where $\mathrm{Hom}_{L_1}(S, W^*)$ is topologically isomorphic to W^*. In this case we shall always assume that the isomorphism is given by the mapping $\alpha : W^* \to \mathrm{Hom}_{L_1}(S, W^*)$ determined by the equations.

$$\langle w, \alpha(w^*)(g)\rangle = \langle g * w, w^*\rangle$$

which are to be valid for every $w \in W$, $w^* \in W^*$, and $g \in S$. The space $M(G)$ of bounded regular Borel measures on G can always be considered as a subspace of $\mathrm{Hom}_{L_1}(S, S)$. This follows at once on noting that if $\mu \in M(G)$ and $g \in S$, then $\mu * g \in S$ and $\|\mu * g\|_S \leq \|\mu\| \, \|g\|_S$ [8, p. 20].

In the following sections we shall use the symbol $\simeq$ to denote "topological isomorphism" whereas $\cong$ will stand for "isometric isomorphism". The end of a proof is indicated by #.

1. THE MAIN THEOREM. The key result needed to establish the theorem alluded to in the first paragraph of the introduction is the following well known theorem from functional analysis.

THEOREM 1. ([6, pp. 277 and 278]). *If* $(V, \|\cdot\|_V)$ *and* $(W, \|\cdot\|_W)$ *are Banach spaces and* $A : V \to W$ *is an injective continuous linear transformation such that* $A(V)$ *is* $\|\cdot\|_W$*-dense in* W, *then the following are equivalent:*

(i) A *is surjective.*

(ii) A^* *is surjective.*

Naturally the symbol A^* denotes the usual adjoint transformation [6, p. 96].

THEOREM 2. *Let* G *be a locally compact Abelian toplogical group, let* S *be a Segal algebra on* G, *and let* W *be an essential* L_1*-convolution module. Then the following are equivalent:*

(i) $S \underset{\sim}{\otimes} W = W$.

(ii) $\mathrm{Hom}_{L_1}(S, W^*) \cong W^*$.

PROOF. If $S \underset{\sim}{\otimes} W$ and W are equal as point sets, then $S \underset{\sim}{\otimes} W \cong W$. This follows at once from the closed graph theorem [6, p. 189] and the fact that the identity mapping $\iota : S \underset{\sim}{\otimes} W \to W$ is continuous. Consequently

$$\begin{aligned}\mathrm{Hom}_{L_1}(S, W^*) &\cong (S \underset{\sim}{\otimes} W)^* \\ &\cong W^* .\end{aligned}$$

Conversely, suppose $\mathrm{Hom}_{L_1}(S, W^*) \cong W^*$; that is, the mapping $\alpha : W^* \to \mathrm{Hom}_{L_1}(S, W^*)$ is a surjective topological isomorphism. Since S is $\|\cdot\|_1$-dense in $L_1(G)$ and W is essential, it follows at once that $\iota(S \underset{\sim}{\otimes} W)$ is $\|\cdot\|_W$-dense in W. More over, we claim that the adjoint transformation $\iota^* : W^* \to (S \underset{\sim}{\otimes} W)^*$ is surjective. Indeed, if

$w = \Sigma_{k=1}^{\infty} g_k * w_k \in S \otimes W$ and $w^* \in W^*$, then

$$\begin{aligned}\langle w, \beta\circ\alpha(w^*)\rangle &= \Sigma_{k=1}^{\infty} \langle g_k * w_k, \beta\circ\alpha(w^*)\rangle \\ &= \Sigma_{k=1}^{\infty} \langle w_k, \alpha(w^*)(g_k)\rangle \\ &= \Sigma_{k=1}^{\infty} \langle g_k * w_k, w^*\rangle \\ &= \langle \iota(w), w^*\rangle \\ &= \langle w, \iota^*(w^*)\rangle.\end{aligned}$$

Thus $\iota^* = \beta\circ\alpha$ is surjective, whence, by Theorem 1, $S \underline{\otimes} W = W$. #

If W is a reflexive essential L_1-module, then it has been proved previously [9, p.473] that $\mathrm{Hom}_{L_1}(L_1(G), W^*) \cong W^*$. The proof appears to depend on the fact that $L_1(G)$ has a bounded approximate identity. A Segal algebra S contains a bounded approximate identity if and only if $S = L_1(G)$ [1, p.552, 8, p. 34].

When $W = C_o(G)$, then in many cases $\mathrm{Hom}_{L_1}(S, M(G)) = \mathrm{Hom}_{L_1}(S, C_o(G)^*) = \mathrm{Hom}_{L_1}(S, S)$. This is, for example, the case if $S = L_1(G)$, $S = A_p(G)$, $1 \leq p < \infty$, or $S = L_2(G)$, G being compact. However, it fails to be so if G is an infinite compact group and $S = L_p(G)$, $2 < p < \infty$. In this case $\mathrm{Hom}_{L_1}(L_p(G), L_p(G))$ is properly contained in $\mathrm{Hom}_{L_1}(L_p(G), M(G)) \cong L_\infty(\hat{G})$ [5, pp. 92 and 110]. This observation explains the additional hypothesis in the next corollary.

COROLLARY. <u>Let</u> G <u>be a locally compact Abelian topological group and let</u> S <u>be a Segal algebra such that</u> $\mathrm{Hom}_{L_1}(S, M(G)) = \mathrm{Hom}_{L_1}(S, S)$. <u>Then the following are equivalent:</u>

(i) $S \otimes C_o(G) = C_o(G)$

(ii) $\mathrm{Hom}_{L_1}(S, S) \simeq M(G)$.

We note in passing that if S is a Segal algebra, then $\mathrm{Hom}_{L_1}(S, M(G)) = \mathrm{Hom}_{L_1}(S, L_1(G))$. This is so because $T \in \mathrm{Hom}_{L_1}(S, M(G))$ if and only if T is continuous and commutes with translation [2,4] and because the measures in $M(G)$ for which translation is norm continuous are precisely the absolutely continuous measures [5, p. 251].

2. APPLICATIONS. We first apply Theorem 2 to obtain necessary and sufficient conditions that $C_o(G)$ be equal to $S \otimes C_o(G)$ for various Segal algebras S.

THEOREM 3. *Let* G *be a nondiscrete locally compact Abelian topological group.*

(i) If $1 \leq p < \infty$, *then* $A_p(G) \otimes C_o(G) = C_o(G)$ *if and only if* G *is noncompact.*

(ii) If $1 < p < \infty$, *then* $(L_1(G) \cap L_p(G)) \otimes C_o(G) = C_o(G)$ *if and only if* G *is noncompact.*

(iii) $(L_1(G) \cap C_o(G)) \otimes C_o(G) = C_o(G)$ *if and only if* G *is noncompact.*

PROOF. Parts (i) and (ii) follow at once from Theorem 2 and the facts that $\mathrm{Hom}_{L_1}(S, M(G)) \simeq M(G)$ if and only if G is noncompact when $S = A_p(G)$, $1 \leq p < \infty$ [5, pp. 204, 207 and 208], or $S = L_1(G) \cap L_p(G)$, $1 < p < \infty$ [5, pp. 79, 92, and 110].

If G is noncompact, then $\mathrm{Hom}_{L_1}(L_1(G) \cap C_0(G), M(G)) \cong M(G)$ [5, p. 80], whereas if G is compact, then $(L_1(G) \cap C_0(G)) \otimes C_0(G) = C(G) \otimes C(G) \subset L_2(G) \otimes L_2(G) = L_1(\hat{G})^{\wedge} \neq C(G)$ [3, pp. 386 and 420]. This proves part (iii).#

If G is discrete, then $S = L_1(G)$ and $C_0(G) = L_1(G) * C_0(G) = L_1(G) \otimes C_0(G)$ [3, p.283].

For compact G we note that an argument using Theorem 1, known descriptions of $\mathrm{Hom}_{L_1}(A_p(G), A_p(G))$, $1 \leq p \leq 2$ [5, p. 207], and the fact that $A_p(G) \subset L_{p'}(G) \subset A_2(G)$, $1 \leq p \leq 2$, $1/p + 1/p' = 1$ [5, p. 209], reveals that $A_p(G) \otimes C(G) = L_{p'}(G) \otimes C(G) = L_1(\hat{G})^{\wedge}$. If $p > 2$, then $L_1(\hat{G})^{\wedge}$ is a proper subset of $A_p(G) \otimes C(G)$ [5, p. 208] . We omit the details.

Since for any locally compact Abelian topological group G, for $1 < p \leq \infty$, and $1/p + 1/p' = 1$, we know that $L_p(G) \cong L_{p'}(G)^*$ and $L_1(G) * L_{p'}(G) = L_{p'}(G)$ [3, p. 272], an application of Theorem 2 immediately yields the known result that $\mathrm{Hom}_{L_1}(L_1(G), L_p(G)) \cong L_p(G)$, $1 < p \leq \infty$ [5, pp. 67 and 68, 9, p. 473]. Pursueing this line of thought a bit further we obtain the following theorem:

THEOREM 4. Let G be a locally compact Abelian topological group and let S be a Segal algebra on G. Then the following are equivalent:

(i) $\mathrm{Hom}_{L_1}(S, L_\infty(G)) \cong L_\infty(G)$.

(ii) $S = L_1(G)$.

PROOF. Since S is an essential L_1-convolution module, it follows that $S = L_1(G) * S = L_1(G) \circledast S = S \circledast L_1(G)$. Consequently, by Theorem 2, $\mathrm{Hom}_{L_1}(S, L_\infty(G)) \simeq L_\infty(G)$ if and only if $S = L_1(G)$.#

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